[This question paper contains 6 printed pages.]

Your Roll No............

Sr. No. of Question Paper : 1114 A

Unique Paper Code : 32351601

Name of the Paper : BMATH 613 – Complex Analysis

Name of the Course : B.Sc. (H) Mathematics

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. All questions are compulsory.

3. Attempt two parts from each question.

1. (a) Sketch the region onto which the sector $r \le 1$, $0 \le \theta \le \pi$ is mapped by the transformation $w = z^2$ and $w = z^3$. (6)

(b) (i) Find the limit of the function $f(z) = \frac{(\bar{z})^2}{z}$ as z tends to 0.

P.T.O.

(ii) Show that $\lim_{z \to 1+\sqrt{3}i} \frac{z^2-2z+4}{z-1-\sqrt{3}i} = 2\sqrt{3}i.$ (3+3=6)

(c) Let u and v denote the real and imaginary components of the function f defined by means of the equations

$$f(z) = \begin{cases} \bar{z}^2/z & \text{when } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}$$

Verify that the Cauchy-Riemann equations are satisfied at the origin z = (0,0). (6)

(d) If $\lim_{z \to z_0} f(z) = F$ and $\lim_{z \to z_0} g(z) = G$, prove that

$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{F}{G}$ if $G \neq 0$. (6)

2. (a) Find the values of z such that

(i) $e^z = 1 + \sqrt{3}i$, (ii) $e^{(2z-1)} = 1$. (3.5+3=6.5)

(b) Show that the roots of the equation $\cos z = 2$ are $z = 2n\pi + i\cosh^{-1}2$ $(n = 0, \pm 1, \pm 2, ...)$. Then express them in the form $z = 2n\pi \pm i\ln(2+\sqrt{3})$ $(n = 0, \pm 1, \pm 2, ...)$. (3.5+3=6.5)

(c) Show that (3.5+3=6.5)

(i) $\log(1+i)^2 = 2Log(1+i)$.

(ii) $\log(-1+\sqrt{3}i) = ln2 + 2\left(n+\frac{1}{3}\right)\pi i \ (n = 0, \pm 1, \pm 2, \ldots)$

(d) Show that $\overline{\exp(iz)} = \exp(i\bar{z})$ if and only if $z = n\pi \ (n = 0, \pm 1, \pm 2, \ldots)$. (6.5)

3. (a) (i) State mean value theorem of integrals.

Does it hold true for complex valued functions? Justify.

(ii) Evaluate $\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta$. (3+3=6)

(b) Parametrize the curves C_1 and C_2, where

C_1: Semicircular path from -1 to 1

C_2: Polygonal path from the vertices $-1, -1+i, 1+i$ and 1

Evaluate $\int_{C_1} z\,dz$ and $\int_{C_2} z\,dz$. (3+3=6)

(c) For an arbitrary smooth curve $C: z = z(t), a \le t \le b$, from a fixed point z_1 to another fixed point z_2, show that the value of the integrals

(i) $\int_{z_1}^{z_2} z\,dz$ and

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(ii) $\int_{z_1}^{z_2} dz$

depend only on the end points of C. (3+3=6)

(d) State ML inequality theorem. Use it to prove that $\left|\int_C \frac{dz}{z^4}\right| \leq 4\sqrt{2}$, where C denotes the line segment from z = i to z = 1. (2+4=6)

4. (a) A function f(z) is continuous on a domain D such that all the integrals of f(z) around closed contours lying entirely in D have the value zero. Prove that f(z) has an antiderivative throughout D. (6.5)

(b) State Cauchy Goursat theorem. Use it to evaluate the integrals

(i) $\int_C \frac{1}{z^2+2z+2}dz$, where C is the unit circle $|z| = 1$

(ii) $\int_C \frac{2z}{z^2+2}dz$, where C is the circle $|z| = 2$

(2.5+2+2=6.5)

(c) State and prove Cauchy Integral Formula. (2+4.5=6.5)

(d) (i) State Liouville's theorem. Is the function $f(z) = \cos z$ bounded? Justify.

(ii) Is it true that 'If $p(z)$ is a polynomial in z then the function $f(z) = 1/p(z)$ can never be an entire function'? Justify (4.5+2=6.5)

5. (a) If a series $\sum_{n=1}^{\infty} z_n$ of complex numbers converges then prove $\lim_{n\to\infty} z_n = 0$. Is the converse true? Justify. (6.5)

(b) Find the integral of $\int_C \frac{\cosh \pi z}{z^3+z}$ where C is the positively oriented circle $|z| = 2$. (6.5)

(c) Find the Taylor series representation for the function $f(z) = \frac{1}{z}$ about the point $z_0 = 2$. Hence prove that $\frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n$ for $|z - 2| < 2$. (6.5)

(d) If a series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges to $f(z)$ at all points interior to some circle $|z - z_0| = R$, then

P.T.O.

prove that it is the Taylor series for the function f(z) in power of $z - z_0$. (6.5)

6. (a) For the given function $f(z) = \frac{z+1}{z^2+9}$ find the poles, order of poles and their corresponding residue. (6)

(b) Write the two Laurent Series in powers of z that represent the function $f(z) = \frac{1}{z+z^3}$ in certain domains and specify those domains. (6)

(c) Suppose that $z_n = x_n + iy_n$, (n = 1,2,3,...) and $S = X + iY$. Then prove that

$$\sum_{n=1}^{\infty} z_n = S \text{ iff } \sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y.$$ (6)

(d) Define residue at infinity for a function f(z). If a function f(z) is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then prove that

$$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right].$$ (6)

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